Journal of Engineering Mathematics 17 (1983) 323–328

On the motion due to sources and sinks distributed on horizontal boundaries in a rotating fluid

S.H. SMITH

Department of Mathematics, University of Toronto, Toronto, Canada M5S 1A1

(Received June 16, 1983)

Summary

We consider the flow of a fluid, contained between infinite discs rotating with constant angular velocity, when subjected to a small, but otherwise general asymmetric distribution of sources and sinks across the discs. It is shown (i) how the mass flux in the horizontal direction is equally shared by the two Ekman layers on the discs, (ii) how all horizontal flows depend only on the difference of the imposed velocities on the discs, and (iii) how in particular situations the geostrophic flow is unbounded.

1. Introduction

In a recent paper Kranenberg [1] considered the flow in a rotating basin due to the withdrawal of fluid placed asymmetrically within the interior of the fluid. He took the sink to have the form of a vertical line extending from the free surface to the bottom of the basin, and his basic conclusion is that a vortex forms along the axis of the sink, which can lead to a counter rotating gyre attached to the far wall. This was a comprehensive study, including transient effects from the onset of the outflow for a fluid with a free surface in a non-linear context.

His paper can be seen as a further stage in the understanding of source-sink flows in a rotating fluid. The initial work was by Barcilon [2] and Hide [3], who considered steady flows in a closed container where the fluid was injected and/or withdrawn from the side walls, and demonstrated the crucial role played by the boundary layers along the side walls in the transport of fluid. Kuo and Veronis [4] examined the influence of a free surface when the Froude number has order unity with the purpose of applying these ideas to an oceanographic context. Also, Johnson [5] presented a detailed theoretical investigation of the flow from a vertical line source to a vertical line sink, set within a rotating cylindrical tank, to support the interpretation given by Hide for one of his experiments.

The intention behind the present note is to briefly consider the situation when the fluid is added and withdrawn on horizontal rather than vertical surfaces. In this regard, Matsuda, Sakurai and Takeda [6] investigated source sink flows in a gas centrifuge when the distribution is on the top and bottom discs. The compressibility makes the calculations much more detailed, and physical conclusions harder to develop. Nevertheless, some general conclusions are given for axisymmetric situation which we redevelop here for an incompressible fluid. The major conclusions of the present study are brought out most clearly through a particular example based on a simple linear model, and although it can be expected that they still have validity as the Rossby number increases, we do restrict the discussion here to bring out these features.

2. General theory

We consider the flow of viscous fluid in a frame of reference which rotates with constant angular velocity Ω about the vertical z-axis. The fluid, with density ρ and viscosity ν, is constrained between infinite discs represented by $z = 0$, h whose motion forms the basic solid body rotation. When (r, θ, z) are the non-dimensional cylindrical co-ordinates of a point in the rotating frame we write the radial, aximuthal and axial velocities as $\epsilon\Omega a\, u(r, \theta, z)$, $\epsilon\Omega a\, v(r, \theta, z)$ and $\epsilon\Omega a\, w(r, \theta, z)$ respectively, where a is a length scale, and ϵ is the small Rossby number which enables the basic linearisation to be taken. When, further, the pressure is defined by $\epsilon\rho\Omega^2 a^2 p(r, \theta, z)$, the equations of motion are

$$u_r + \frac{1}{r}u + \frac{1}{r}v_\theta + w_z = 0, \tag{2.1}$$

$$-2v = -p_r + E\left(u_{rr} + \frac{1}{r}u_r - \frac{1}{r^2}u + \frac{1}{r^2}u_{\theta\theta} + u_{zz} - \frac{2}{r^2}v_\theta\right), \tag{2.2}$$

$$2u = -\frac{1}{r}p_\theta + E\left(v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v + \frac{1}{r^2}v_{\theta\theta} + v_{zz} + \frac{2}{r^2}u_\theta\right), \tag{2.3}$$

$$0 = -p_z + E\left(w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\theta\theta} + w_{zz}\right), \tag{2.4}$$

E is the Ekman number for the flow, with $E = \nu/\Omega a^2$, and is taken to be small in all which follows.

There is a combination of imposed inflow and outflow across the surfaces of the two discs, with the restriction that there is no net increase of mass. Hence we impose the boundary conditions

$$u = v = 0 \text{ on } z = 0, h, \tag{2.5}$$

$$w = E^{1/2}W_0(r, \theta) \text{ on } z = 0, \qquad w = E^{1/2}W_1(r, \theta) \text{ on } z = h, \tag{2.6}$$

where

$$\int_0^\infty r\mathrm{d}r \int_{-\pi}^{\pi} W_0 \mathrm{d}\theta = \int_0^\infty r\mathrm{d}r \int_{-\pi}^{\pi} W_1 \mathrm{d}\theta.$$

The equations (2.1)–(2.4) cannot be solved generally for distributions W_0, W_1, but asymptotic methods for $E \ll 1$ can describe the basic features. To begin, we summarize the analysis for the Ekman layers, there is only a slight modification of previous work (cf. Jacobs [8], Kuo and Veronis [4]).

When $z = E^{1/2}\zeta$, we write $w = E^{1/2}W(r, \theta, \zeta)$ for the layer on the lower disc to provide the equations

$$u_r + \frac{1}{r}u + \frac{1}{r}v_\theta + W_\zeta = 0, \qquad -2v = -p_r + u_{\zeta\zeta},$$

$$2u = -\frac{1}{r}p_\theta + v_{\zeta\zeta}, \qquad p_\zeta = 0,$$

the last equation showing that $p \equiv p(r, \theta)$ throughout the fluid. The solutions for u and v which are zero on the disc are given by

$$2u = -p_r\,\mathrm{e}^{-\zeta}\sin\zeta - r^{-1}p_\theta(1 - \mathrm{e}^{-\zeta}\cos\zeta), \tag{2.7a}$$

$$2v = p_r(1 - \mathrm{e}^{-\zeta}\cos\zeta) - r^{-1}p_\theta\,\mathrm{e}^{-\zeta}\sin\zeta. \tag{2.7b}$$

The continuity equation then shows

$$W = W_\mathrm{I} - \tfrac{1}{4}\nabla^2 p\,\mathrm{e}^{-\zeta}(\cos\zeta + \sin\zeta), \qquad \zeta = \mathrm{O}(1), \tag{2.8}$$

where $W_\mathrm{I}(r, \theta)$ is the interior axial velocity. For the Ekman layer on the upper disc, where we define $h - z = E^{1/2}\xi$, the expressions (2.7) are changed only by replacing ζ by ξ, and (2.8) becomes

$$W = W_\mathrm{I} + \tfrac{1}{4}\nabla^2 p\,\mathrm{e}^{-\xi}(\cos\xi + \sin\xi), \qquad \xi = \mathrm{O}(1). \tag{2.9}$$

Consequently, when $W = W_0$ on $z = 0$, $W = W_1$ on $z = h$, it follows that

$$W_\mathrm{I} = \tfrac{1}{2}(W_0 + W_1), \tag{2.10}$$

$$\nabla^2 p = 2(W_1 - W_0). \tag{2.11}$$

Although these results appear to be unsurprising, a number of conclusions can be drawn immediately from them; the Poisson equation for $p(r, \theta)$ is a generalization of the Laplace equation given by Greenspan [7] when there is no flow imposed across the horizontal discs.

The equation for p depends only on the difference between W_0 and W_1, and so the addition of an inflow over any finite domain of one disc, plus an equal outflow over the same domain of the other disc, will have no effect on the horizontal flow in the interior or in the Ekman layers. The extra axial velocity on the discs will just be added to the axial velocity W_I through (2.10), and be present in both the Ekman layers from (2.8), (2.9). Consequently, there is an insensitivity of the horizontal flow to the precise conditions on the discs.

Secondly, the flow parallel to the discs in each of the Ekman layers is identical. Now there is no horizontal transport of mass of $\mathrm{O}(E^{1/2})$ in the geostrophic interior, with the flux in this direction taking place predominantly within the Ekman layers. Hence, this transport must be equally balanced between the two layers to $\mathrm{O}(E^{1/2})$, even in situations where both inflow and outflow is from only one of the discs.

Finally, in any domain in the r, θ plane where $\nabla^2 p = 0$ we can introduce the harmonic function $s(r, \theta)$ conjugate to $p(r, \theta)$ by

$$s_r = -r^{-1}p_\theta = 2u_\mathrm{I}, \qquad r^{-1}s_\theta = p_r = 2v_\mathrm{I}, \tag{2.12}$$

where u_I, v_I are the interior velocities. From potential theory, the curves $s =$ constant are orthogonal to those where $p =$ constant, and so the mass flux within the Ekman layers takes place along the set of curves orthogonal to those where $p(r, \theta) =$ constant.

For any particular situation, to gain a description of the interior flow, and therefore of the Ekman layer behaviour, the mathematical problem reduces to solving the Poisson equation (2.11) subject to certain boundary conditions. Here we take both the inflow and outflow to be defined in such a way that the right hand side of (2.11) is nonzero in some finite domain D (possibly made up of distinct, connected sub-domains D_k), and be zero elsewhere. Along the boundary of each sub-domain D_k there will, in general, be both one-third and one-quarter Stewartson layers, but the behaviour within these is not sufficiently different from that described by Greenspan [7] to detail here. No vertical $\mathrm{O}(E^{1/2})$ transport takes place in these Stewartson layers; this is concentrated within the columns D. Now $p(r, \theta)$ is effectively a stream function for the interior flow, and so it follows that $p =$ constant forms the condition on the boundaries of D_k for the solution of the Poisson equation. Further, continuing to follow Greenspan's arguments, it is seen from the fact that there is no mass flux across these Stewartson layers, that the line integral

$$\int_{C_k} \frac{\partial p}{\partial n}\,\mathrm{d}s = 0,$$

where $\partial/\partial n$ represents the normal derivative for points on the curve C_k, which is the closed curve boundary of the domain D_k; when the flow is axisymmetric this condition requires that the angular velocity v is continuous across the curve C_k.

3. Particular example

We present just one example here which, it is believed, fully illustrates the results of the previous section. Solving the Poisson equation in particular domains is a straightforward mathematical exercise, but even this can be avoided by the intuitive reader on the basis of the following simple, illustrative example.

We set

$$W_0 = \begin{Bmatrix} -2/c_0^2 & \text{inside } D_0 \\ 0 & \text{elsewhere} \end{Bmatrix}, \qquad W_1 = \begin{Bmatrix} -2/c_1^2 & \text{inside } D_1 \\ 0 & \text{elsewhere} \end{Bmatrix}, \tag{3.1}$$

where D_0 and D_1 are two circles, with radii c_0 and c_1 respectively, which have no points in common. Inside the domain D_0 the interior flow will satisfy the Poisson equation $\nabla^2 p = 4/c_0^2$, with $p =$ constant on the boundary; consequently there is relative solid body rotation with the angular velocity difference given by $1/c_0^2$. A similar conclusion follows for the behaviour inside D_1. Outside these circular columns the flow is irrotational, and the streamlines $p =$ constant form a system of coaxial circles which include (and are defined by) D_0 and D_1; there is a radical axis which divides the interior into two distinct

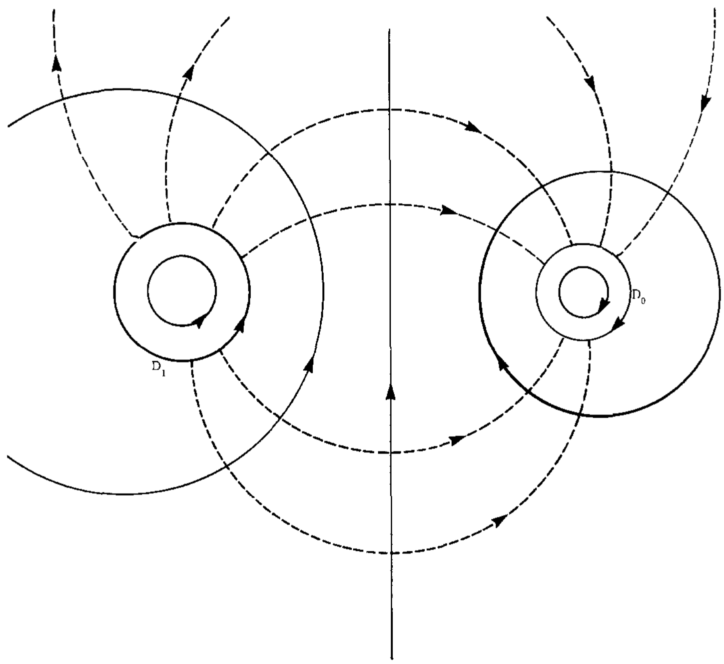

Figure 1. Full lines represent the streamlines for the geostrophic flow. Dotted lines represent the direction of mass transport in the Ekman layers.

regions. The orthogonal trajectories represent the streamlines for the flow of mass within the Ekman layers; these curves are also circles which pass through the radical points of the first system. Exactly half the fluid is transmitted through each of the Ekman layers on the separate discs. The streamlines are sketched in Fig. 1.

The interesting feature to notice here is that the geostrophic flow induced in the interior is not confined, even though the distribution of sources and sinks is bounded; this type of behaviour has not been observed before and does not seem to be possible in axisymmetric situations. The velocities decay as $O(r^{-2})$ when $r \to \infty$.

We can extend the above results to the situation where $W_0 = 0$ everywhere, plus $W_1 = 2/c_0^2$ inside D_0, $W_1 = -2/c_1^2$ inside D_1, and $W_1 = 0$ elsewhere. These conditions are the same as (3.1) except for the addition of the constant velocity $2/c_0^2$ throughout the column D_0. The mass is now both injected and extracted on the upper disc alone. However, there is no change in the flow in any horizontal plane, and the streamlines of Fig. 1 are still valid; half the mass flux still takes place in the Ekman layer on the lower disc.

To conclude, we note that the interior flow described by Kranenberg [1] can be reproduced by a particular distribution of sources and sinks on the discs, as predicted from his experimental results. If the conditions (3.1) are set, except that now the circle D_0 is completely contained within D_1, then the behaviour described by Kranenberg follows in the limit as $c_0 \to 0$. The more general problem, with finite c_0, can be solved using bipolar co-ordinates for the domain between D_0 and D_1, but the details take away from the basic purpose of the present note and are not given here.

Acknowledgement

The author wishes to thank the Natural Sciences and Engineering Research Council of Canada for a grant in aid of research during the time this work was completed.

References

[1] C. Kranenberg, Sink flow in a rotating basin, *J. Fluid Mech.* 94 (1979) 65–81.
[2] V. Barcilon, On the motion due to sources and sinks distributed along the vertical boundary of a rotating fluid, *J. Fluid Mech.* 27 (1967) 551–560.
[3] R. Hide, On source-sink flows in a rotating fluid, *J. Fluid Mech.* 32 (1968) 737–764.
[4] H.-H. Kuo and G. Veronis, The source-sink flow in a rotating system and its oceanic analogy, *J. Fluid Mech.* 45 (1971) 441–474.
[5] J.A. Johnson, A source-sink flow in a rotating fluid, *Proc. Camb. Phil. Soc.* 75 (1974) 269–282.
[6] T. Matsuda, T. Sakurai and H. Takeda, Source-sink flow in a gas centrifuge, *J. Fluid Mech.* 69 (1975) 197–208.
[7] H.P. Greenspan, *The theory of rotating fluids*, Cambridge University Press (1968).
[8] S.J. Jacobs, The Taylor column problem, *J. Fluid Mech.* 20 (1964) 581–591.